Test 3 TF, 7 MC questions.

2 possible answers

4 possible answers.

(a) $2^3 \cdot 4^7$ ways to fill out test.

(b) 8 correct, 2 wrong.

- 2 TF wrong $\binom{3}{2} \cdot 1 \cdot 1$
- 2 MC wrong $\binom{7}{2}$ — to choose questions $\cdot\, 3 \cdot 3$
- 1 TF, 1 MC wrong

9 correct, 1 wrong

- 1 TF wrong
- 1 MC wrong.

all correct — 1 way

Last Time

Pascal Identity.

$$\boxed{\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}}$$

1
1 1
1 2 1 ($n=3$)
1 3 3 1
1 4 6 4 1 ($n=4$, $k=2$)

Can we prove it?

Proof: Method 1

$$LHS = \binom{n+1}{k} = \frac{(n+1)!}{k!\,\underbrace{(n+1-k)!}_{\text{cancel}}} = \frac{(n+1)(n)(n-1)\cdots(n-k+2)}{k(k-1)(k-2)\cdots 1}$$

$$RHS = \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k-1)!\,(n-k+1)!}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 1} + \frac{k\,n(n-1)(n-2)\cdots(n-k+2)}{k(k-1)(k-2)\cdots 1}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+2)(n-k+1)}{k!} + \frac{n(n-1)(n-2)\cdots(n-k+2)k}{k!}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+2)(n-k+1) + n(n-1)(n-2)\cdots(n-k+2)k}{k!}$$

$$= \frac{n(n-1)(n-2)\cdots(n-k+2)\,\overbrace{[(n-k+1)+k]}^{n+1}}{k!}$$

$$= \frac{(n+1)n(n-1)\cdots(n-k+2)}{k!}$$

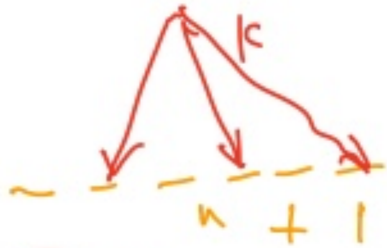
$$= \binom{n+1}{k} = LHS. \quad \blacksquare$$

Algebraic proof

To prove: $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$

$\boxed{\text{Method 2}}$

$\binom{n+1}{k}$ = # of ways to choose k objects from a set of $(n+1)$ objects



= (# of ways to choose k objects from the first n in my set)

+ (# of ways to choose $(k-1)$ from the first n + the extra 1)

$$= \binom{n}{k} + \binom{n}{k-1}. \quad \blacksquare$$

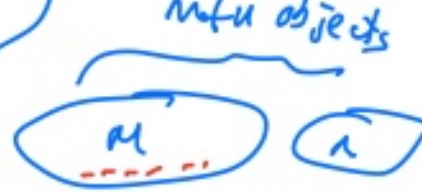
Combinatorial proof

Van der Monde Identity $\displaystyle\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$

where $0 \le r \le m+n$, $m+n$ are positive integers.

Combinatorial Proof:

$\binom{m+n}{r}$ = # of ways to choose r objects from a set of $m+n$ objects



= (# of ways to choose r from 1^{st} m + 0 from last n) + (# of ways to choose $(r-1)$ from 1^{st} m + 1 from ~~2nd~~ last n) +

... + (# of ways to choose 0 from 1^{st} m & r from last n)

$$= \binom{m}{r}\binom{n}{0} + \binom{m}{r-1}\binom{n}{1} + \binom{m}{r-2}\binom{n}{2} + \cdots$$

$$\cdots + \binom{m}{0}\binom{n}{r}.$$

$$= \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

eg $\binom{5}{2} = \binom{2}{2}\binom{3}{0} + \binom{2}{1}\binom{3}{1} + \binom{2}{0}\binom{3}{2}$

$5 = 2+3$

$LHS = \binom{5}{2} = \frac{5.4}{2.1} = 10$

$RHS = 1\cdot 1 + 2\cdot 3 + 1\cdot 3 = 1+6+3 = 10.$

Generalization

$$\binom{n_1 + \cdots + n_r}{m} = \sum_{k_1+k_2+\cdots+k_r=m} \binom{n_1}{k_1}\binom{n_2}{k_2}\cdots\binom{n_r}{k_r}$$

Examples of identities from Binomial Theorem $(A+B)^n = \sum_{k=0}^{n} \binom{n}{k} A^k B^{n-k}$

$\longrightarrow$

Pick different #s for A & B to get different identities

eg $A=2$, $B=1$

$$(2+1)^n = \sum_{k=0}^{n} \binom{n}{k} 2^k 1^{n-k}$$

$$\Rightarrow \quad 3^n = \sum_{k=0}^{n} 2^k \binom{n}{k}$$

eg $$3^3 = \sum_{k=0}^{3} 2^k \binom{3}{k} = 1 \cdot 1 + 2 \cdot 3 + 4 \cdot 3 + 8 \cdot 1$$

$$= 1 + 6 + 12 + 8 = 27. \checkmark$$